Closing Thu: TN 3

Final: Sat, June 3rd, 5:00-7:50pm, KANE 130

Closing Tue: TN 4

Closing next Thu: TN 5 (Last HW)

$$\frac{1}{0!}f(b) + \frac{1}{1!}f'(b)(x-b) + \frac{1}{2!}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

Taylor's Inequality (error bound):

on a given interval [a,b],

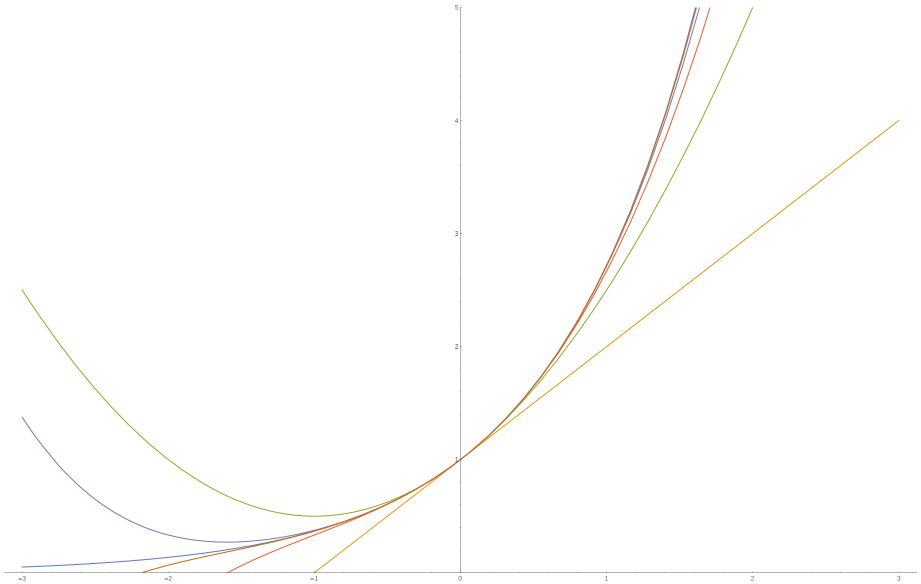
if $|f^{(n+1)}(x)| \leq M$, then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$$

Entry Task:

Find the 9th Taylor polynomial for $f(x) = e^x$ based at $b = 0$, and give an error bound on the interval $[-2,2]$.

$f(x) = e^x$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ are shown:



Entry Task: Again consider,

$$f(x) = e^x \text{ based at } b = 0$$

Find the first value of *n* when Taylor's inequality gives an error less than 0.0001 on $[-2,2]$.

TN 4: Taylor Series

Def'n: The **Taylor Series** for f(x) based at b is defined by

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$$

If the limit exists at a particular value of *x*, then we say the series **converges** at *x*.
(i.e. the error goes to zero at x)
Otherwise, we say it **diverges** at x.

The **open interval of convergence** gives the largest open interval of values over which the series converges.

Note: If

$$\lim_{n\to\infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then *x* is in the open interval of convergence.

A few patterns we know:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

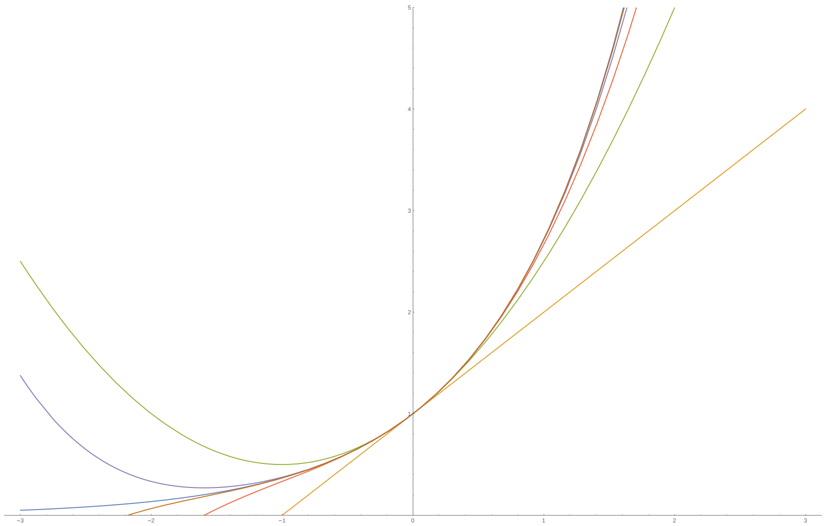
$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

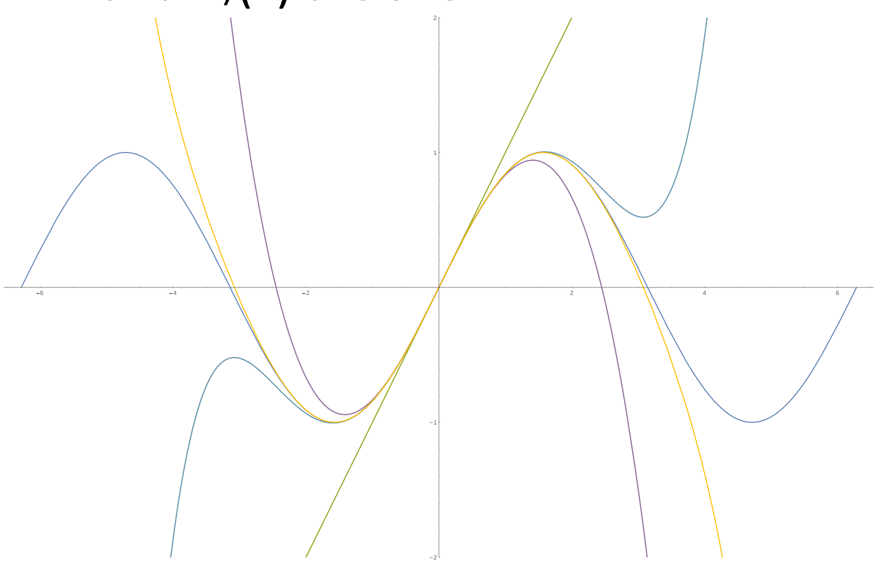
These converge for ALL values of *x*. So the **open interval of convergence** for each series above is (-∞,∞)

Visuals of Taylor Polynomials:

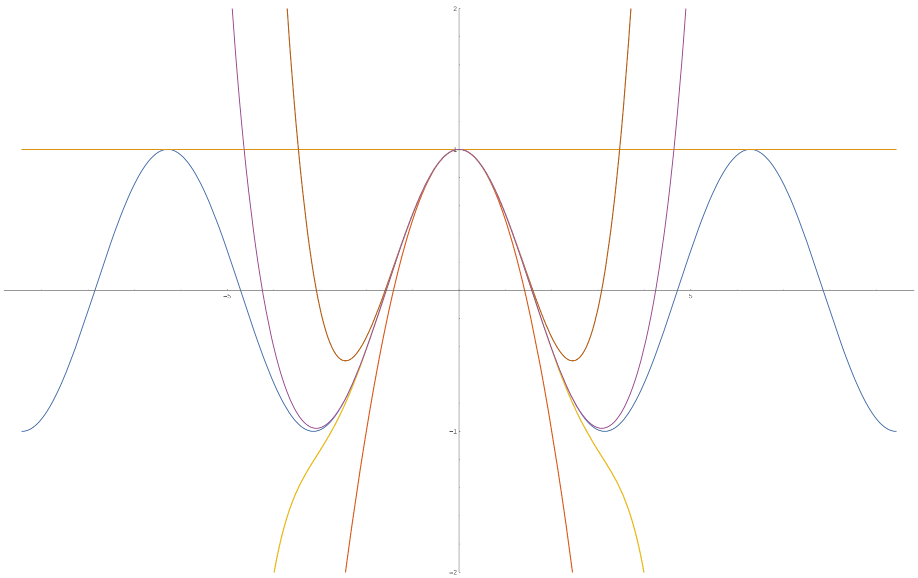
1. $f(x) = e^x$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ are shown:



2. $f(x) = \sin(x)$ as well as $T_1(x)$, $T_3(x)$, $T_5(x)$, and $T_7(x)$ are shown:



3. $f(x) = \cos(x)$ as well as $T_1(x)$, $T_2(x)$, $T_4(x)$, $T_6(x)$, and $T_8(x)$ are shown:



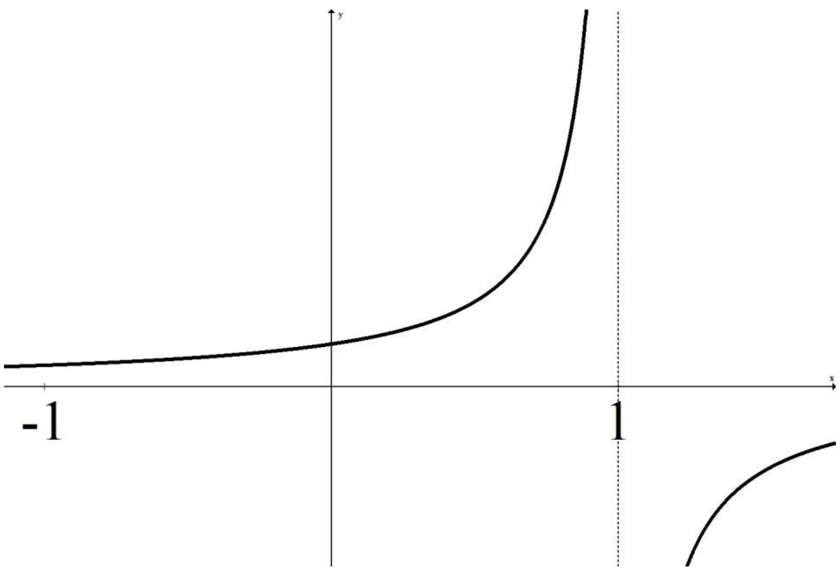
Now consider $f(x) = \frac{1}{1-x}$ based at x = 0.

Find the 10^{th} Taylor polynomial.

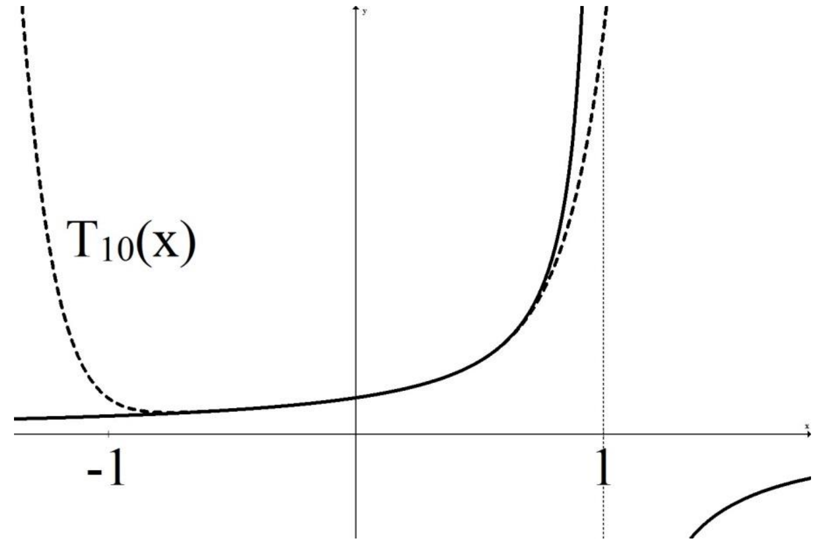
What is the error bound on [-1/2,1/2]?

What is the error bound on [-2,2]?

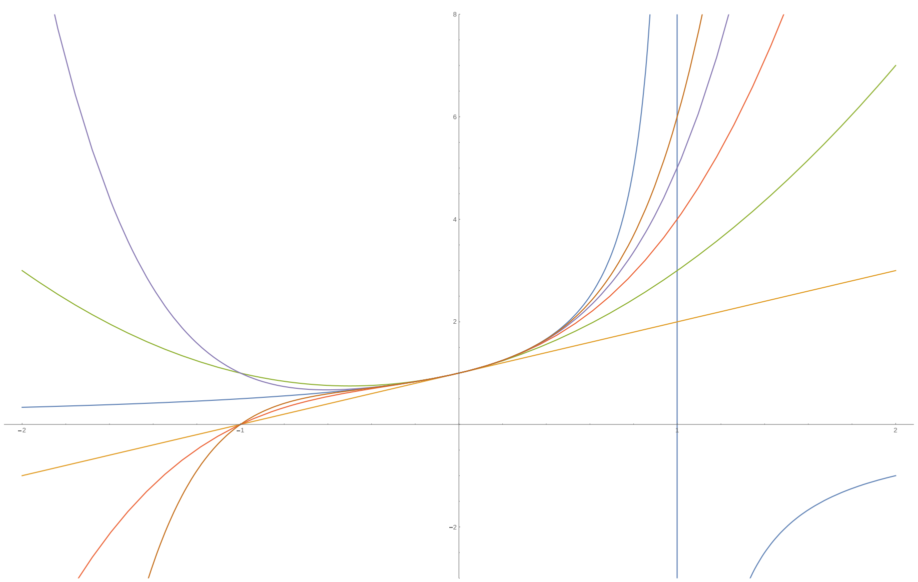
Graph of y = 1/(1-x):



Graph of y = 1/(1-x) and $T_{10}(x)$:



f(x) = 1/(1-x) as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, and $T_5(x)$ are shown:



We will find all the following, and for these they converge for **-1 < x < 1**. In other words, **the open interval of convergence for these series is: -1 < x < 1.**

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$$

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1}$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$